Advantages and dangers on utilizing GeoGebra Automated Reasoning Tools

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August 17, 2018

Abstract

GeoGebra Automated Reasoning Tools is a module of the dynamic mathematics software GeoGebra that combines dynamic geometry and computer algebra to exploit modern methods in formalizing and proving conjectures based on algebraic geometry.
In this contribution some unequivocal results on this novel tool will be addressed, and also a list of challenges on the educational use are given.

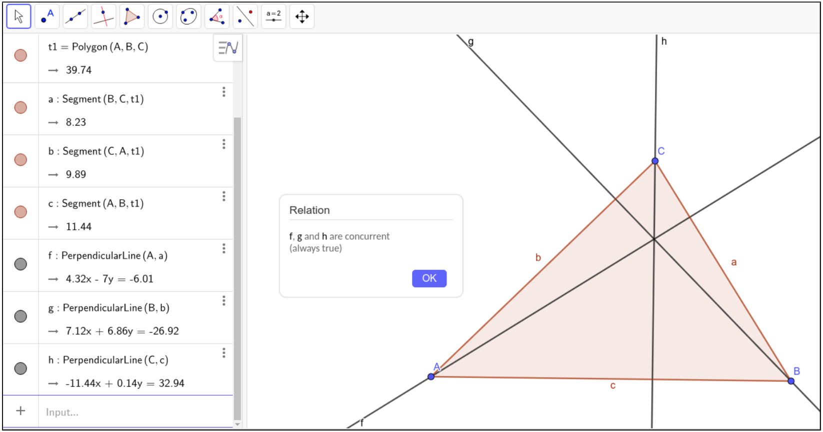
GeoGebra Automated Reasoning Tools (ART)

an embedded module of the free, popular software tool GeoGebra

- started in the beginning of the 2010's
- its foundations are included in Chou's revolutionary book *Mechanical geometry theorem proving*, and in works of former authors including Wu, Buchberger, Tarski and Hilbert
- exploits advantages of planar dynamic geometry visualizations, and adds symbolic checks of user-initiated conjectures in an intuitive way
  - the `Relation` tool can perform a symbolic check of numerical perceptions of typical geometric properties between objects (parallelism, perpendicularity, equality, concurrency etc.)
  - commands like `LocusEquation` and `Envelope` can obtain dynamic locus curves based on pure symbolic computations
    - low-level commands `Prove` and `ProveDetails` are provided for researchers
- a relatively new tool $\rightarrow$ there is not enough feedback to confirm or confute the confidence/doubt on its use cases

An introductory example (Relation)

Altitudes of a triangle are concurrent



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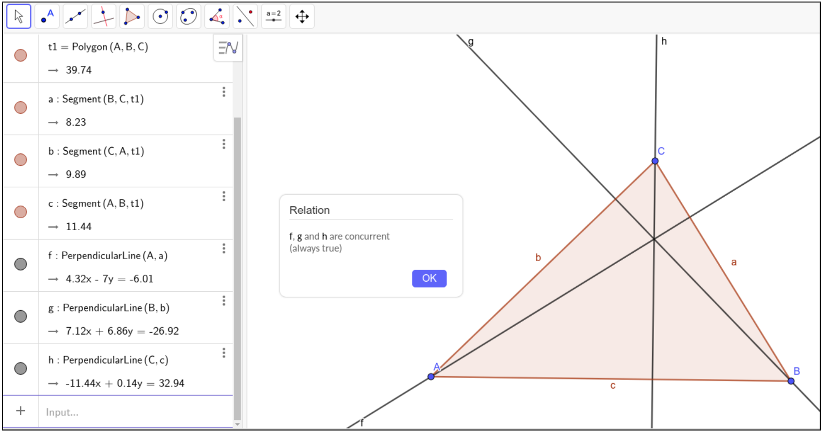

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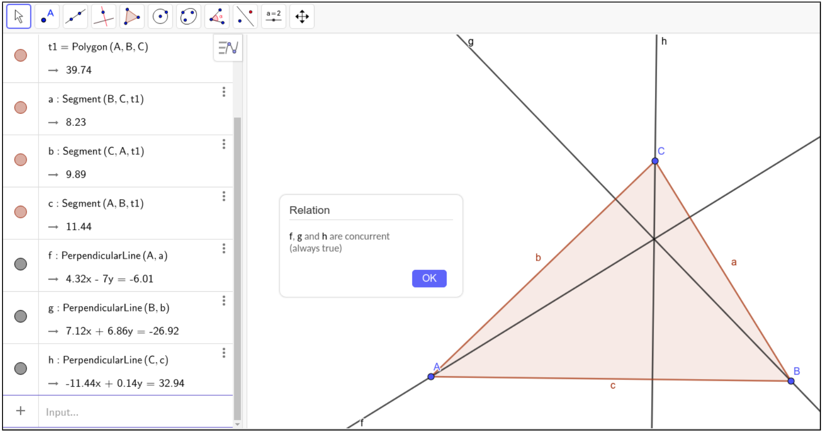
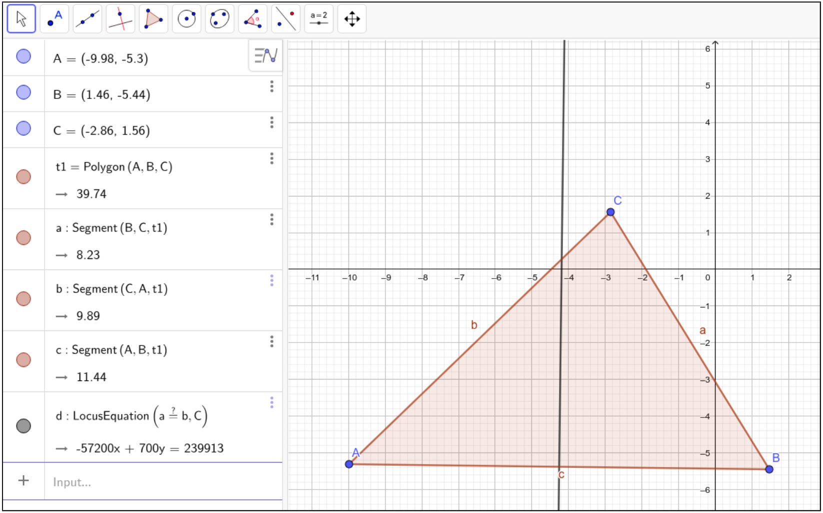


`Relation(`$\{f, g, h\}$`)`

See also `https://www.geogebra.org/m/bv8u4xbv` for a click-only solution.

An introductory example (LocusEquation)

The locus of all points equidistant from two given points



`LocusEquation(a==b, C)`

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- On continuous dragging, can we find a counterexample in the graphical/algebraic representations? Not really.
- Let us state a conjecture! If $A \neq B$, then for all points C' of the perpendicular bisector of AB, $|AC'| = |BC'|$.

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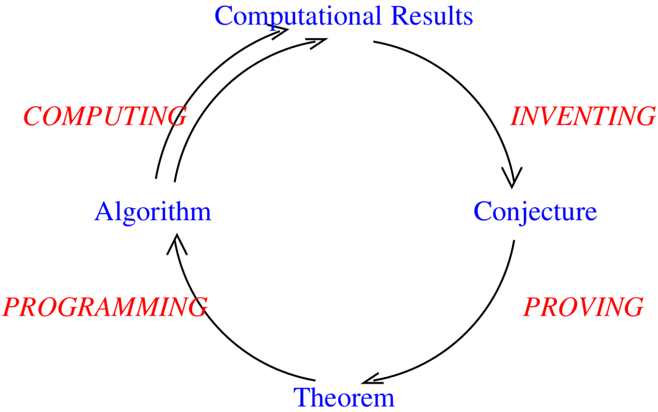
- Non-linear outputs may introduce further challenges (for circles: completing the square, for higher degree outputs: factorization over $\mathbb{Z}$ or $\mathbb{C}$).
- Philippe R. Richard considers this method a *mechanical proof* (since, in general we accept Bézout's algebraic geometry theorem without a proof).

Suggestions on classroom use of this *instrument*

Buchberger's *creativity spiral*



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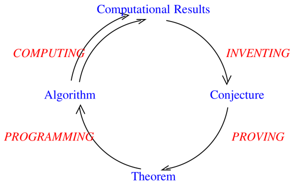
1. Some (random or planned) *computations* are performed with GeoGebra. We get an implicit locus (via `LocusEquation`).
2. A *conjecture* for the output curve is made by the student.
3. The conjecture is checked by the `Relation` tool or command in GeoGebra. We can accept this result without further verification reliably. So we have a *theorem*.

   (Occasionally the proof can be worked out by paper and pencil as well.)
4. "Programming": designing new applets based on new *algorithms* that use the theorem.

1. The theorem can be generalized or modified by plotting further implicit loci with GeoGebra—as further experiments for the student (controlled by the teacher or not).



Then, the process continues from the 2. step again.

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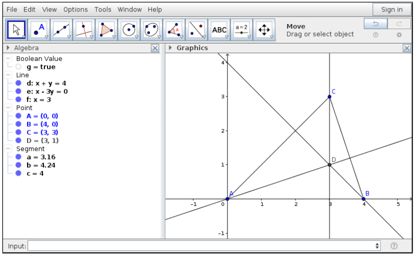

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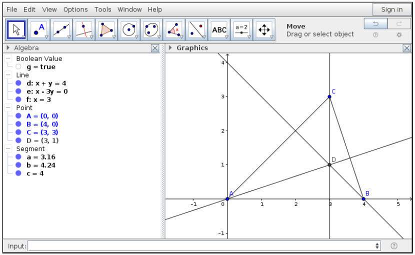

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The numerical check is correct.

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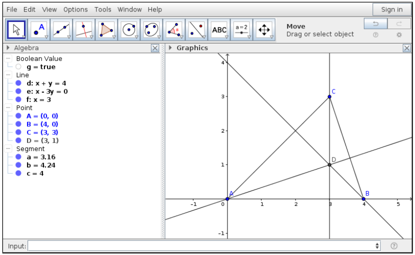

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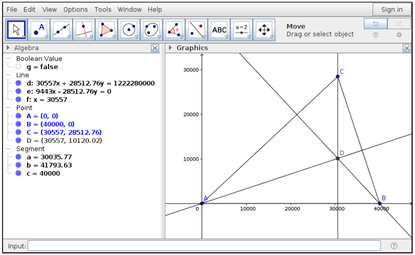

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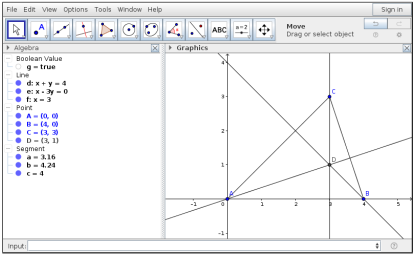

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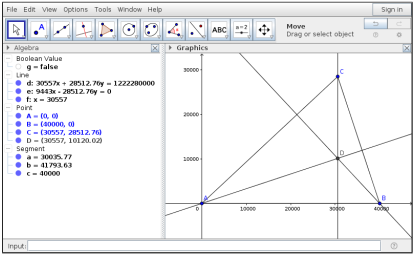

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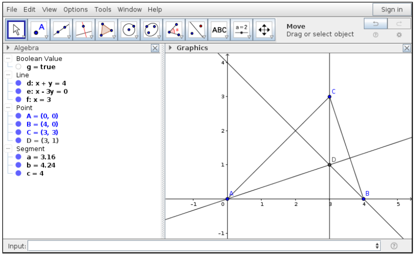

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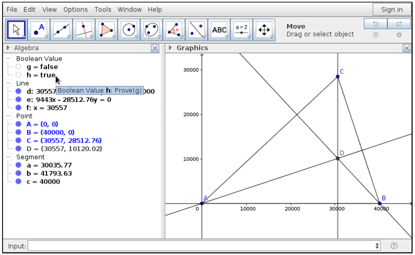

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Wrong numerical check, but the symbolic one is correct!

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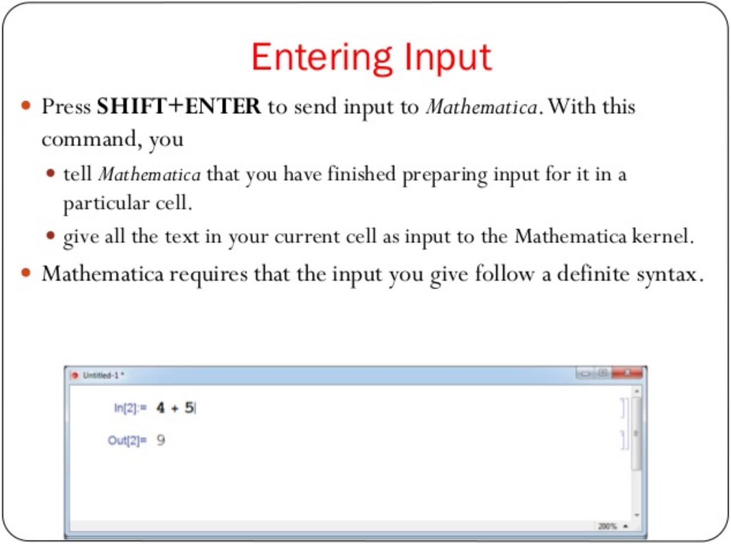

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"Do we really need the solution? Couldn't we just enjoy the problem for now?"

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