Formal Languages for Mathematics

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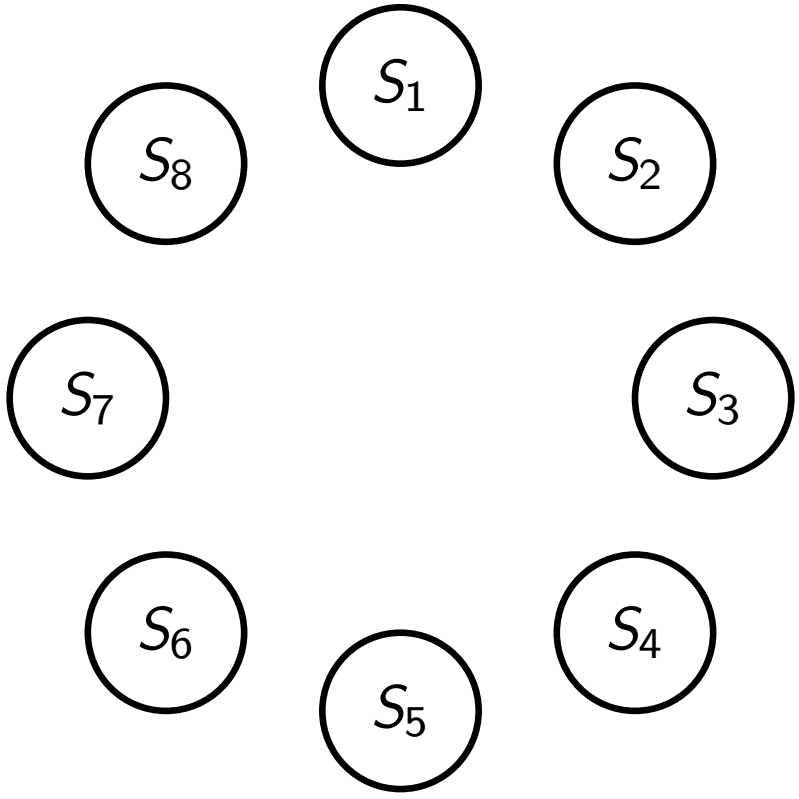
- Most formalisms fail to capture the flexible nature of in-the-wild chalk-and-blackboard mathematics.
- Type systems are ubiquitous, but often lack key features to keep them decidable
- Many systems are also either flexible *or* have good tool, support, not both.

Motivation (2)



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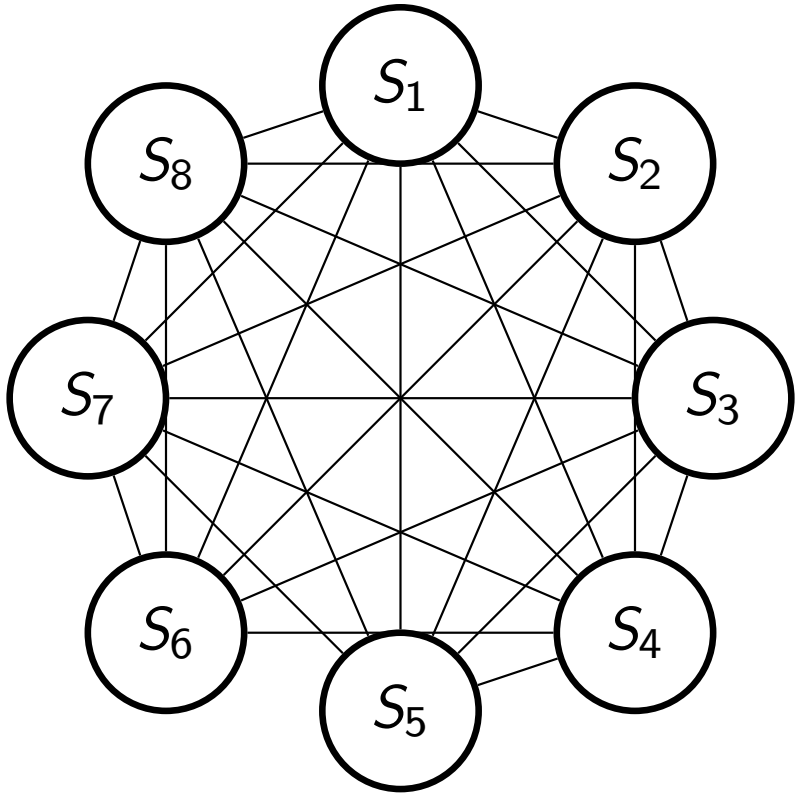

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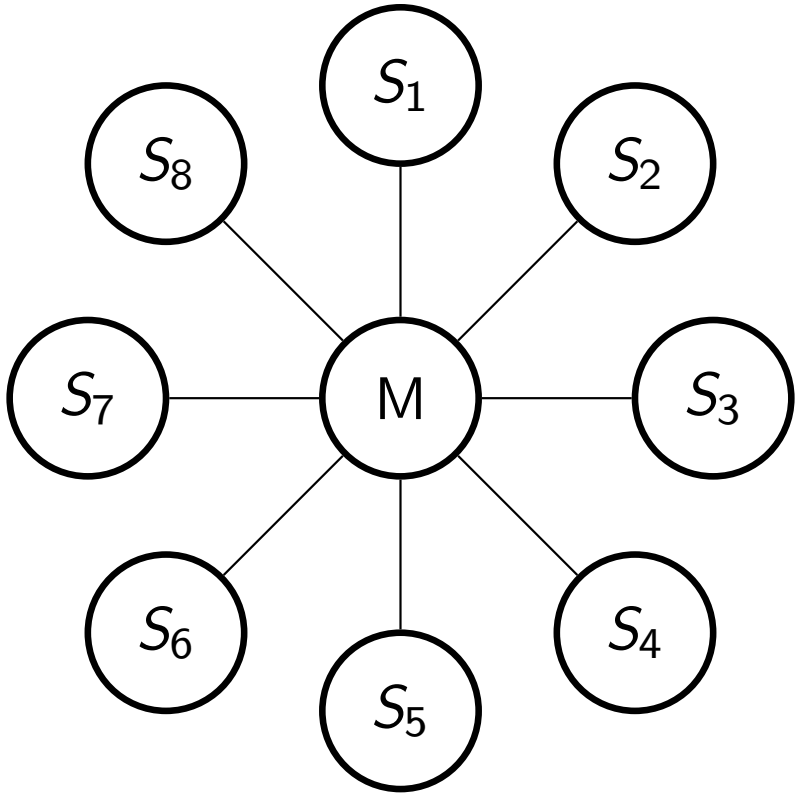

Context & Preliminaries

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One of the goals in developing IMPS was to create a mathematical system that gave computational support to mathematical techniques common among actual mathematicians.

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- Partial functions, subsorts, definite description operator

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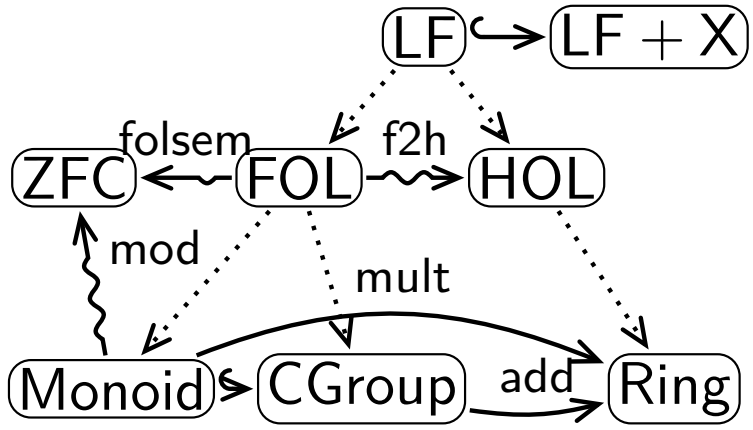
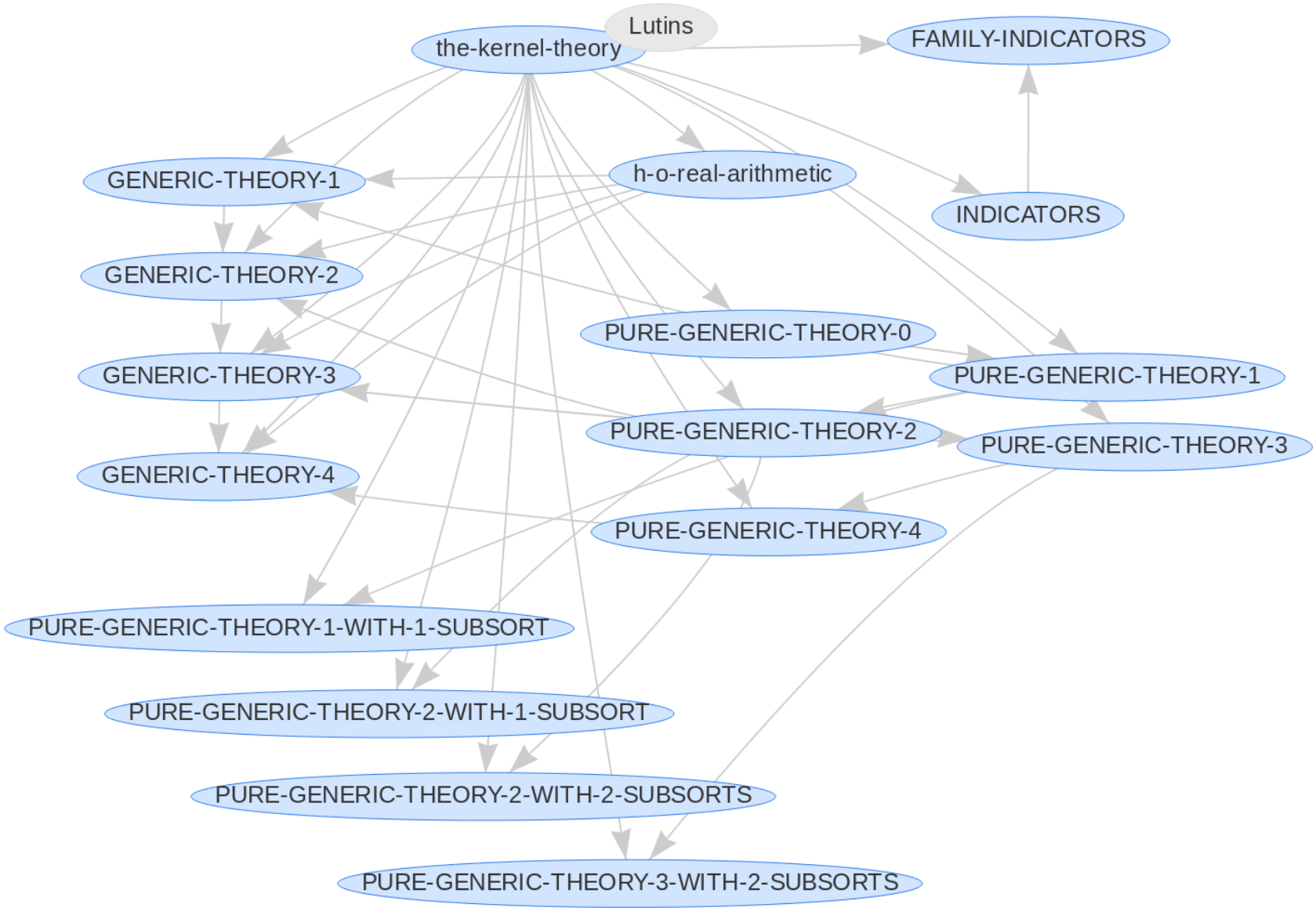
OMDoc/MMT brings with it three distinct levels for expression of (both formal and informal) mathematical knowledge, structurally similar to `IMPS`:

- **Object Level**
  Expressions (e.g. terms and formulae) expressed in OpenMath.
- **Declaration Level**
  Constants (functions, types, judgements) with an optional (object-level) type and/or definition.
- **Module Level**
  Theories and Views; sets of declarations that inhabit a common namespace and context.

MMT

The OMDoc/MMT language is used by the MMT system, which provides an API to handle OMDoc/MMT content and services such as type checking, rewriting of expressions and computation, as well as notation-based presentation of OMDoc/MMT content and a general infrastructure for inspecting and browsing libraries.




Lutins
the-kernel-theory
FAMILY-INDICATORS
GENERIC-THEORY-1
h-o-real-arithmetic
INDICATORS
GENERIC-THEORY-2
PURE-GENERIC-THEORY-0
GENERIC-THEORY-3
PURE-GENERIC-THEORY-1
PURE-GENERIC-THEORY-2
PURE-GENERIC-THEORY-3
GENERIC-THEORY-4
PURE-GENERIC-THEORY-4
PURE-GENERIC-THEORY-1-WITH-1-SUBSORT
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Goals

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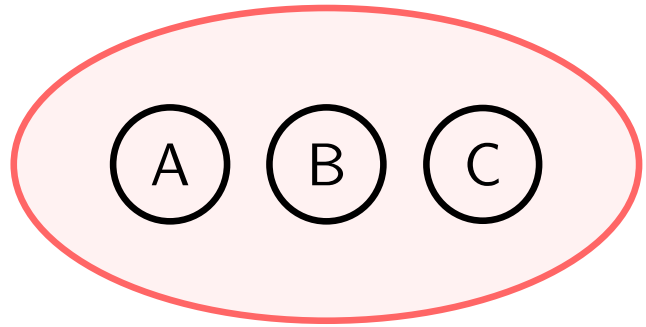

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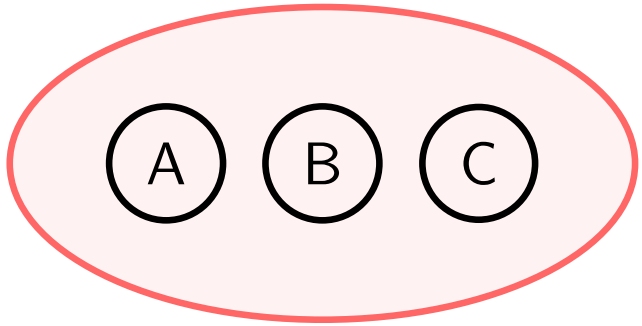

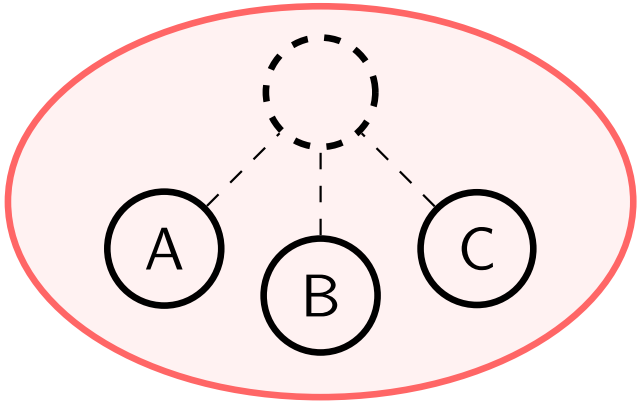

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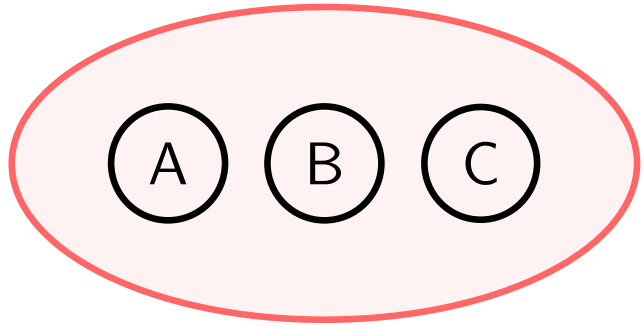

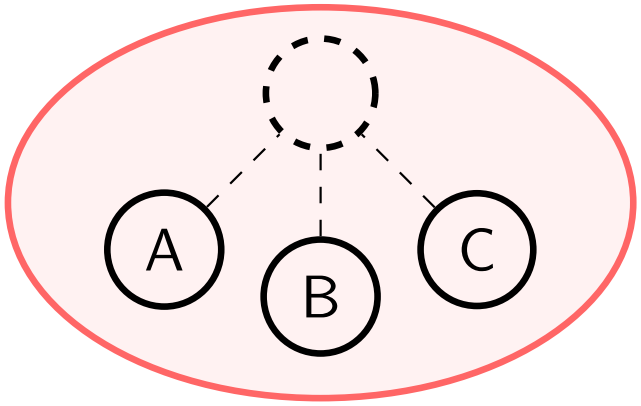

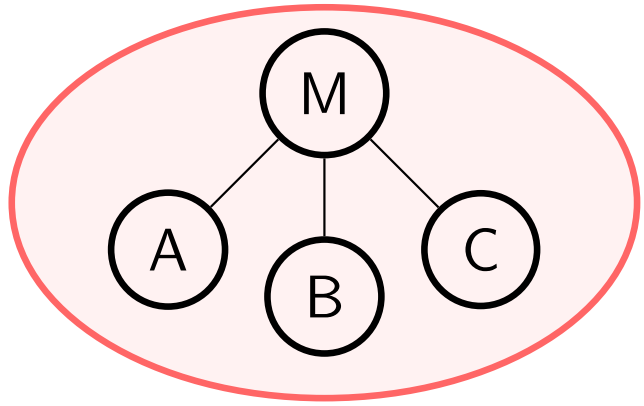

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Finding a treatment of undefinedness that integrates well the MMT system.

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Partial valuation for terms, total valuation for formulas and the divide into two concrete kinds of all mathematical expressions seem a bit inelegant.

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$$\Gamma \vdash t = t' \qquad \Gamma \vdash t : A$$

$$\Gamma \vdash t \downarrow \qquad \Gamma \vdash ? : A$$

Efforts
(planned)

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- Computation Systems
  (e.g. GAP, Sage) *Insufficient treatment of undefinedness.*

Efforts (2)

Import/Export (to and from MMT) for HOL Light.

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Concretely: MMT as of today has support for annotating declarations of equalities so that the simplifier is able to use them as additional rules at runtime.

This process needs to be extended to also allow the introduction of rules with *premises* and the automated checking of the same.

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It also might lead to some insights if an approach like this, with the flexibility it brings to the table, is suitable for formalisation.