Progress in the formalization of Matiyasevich's theorem in the Mizar system

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Hilbert's Tenth Problem

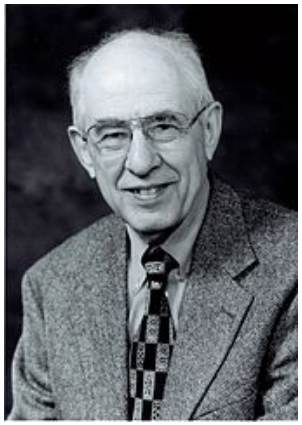
Hilbert's Question

Is there an algorithm which can determine whether or not an arbitrary polynomial equation in several variables has solutions in integers?

Modern formulation

There exists a program taking coefficients of a polynomial equation as input and producing *yes* or *no* answer to the question: *Are there integer solutions?*

The Answer

Martin Davis, Julia Robinson, Yuri Matiyasevich(b. 1947), Hilary Putnam

Negative solution of Hilbert's tenth problem (≈1949–1970)

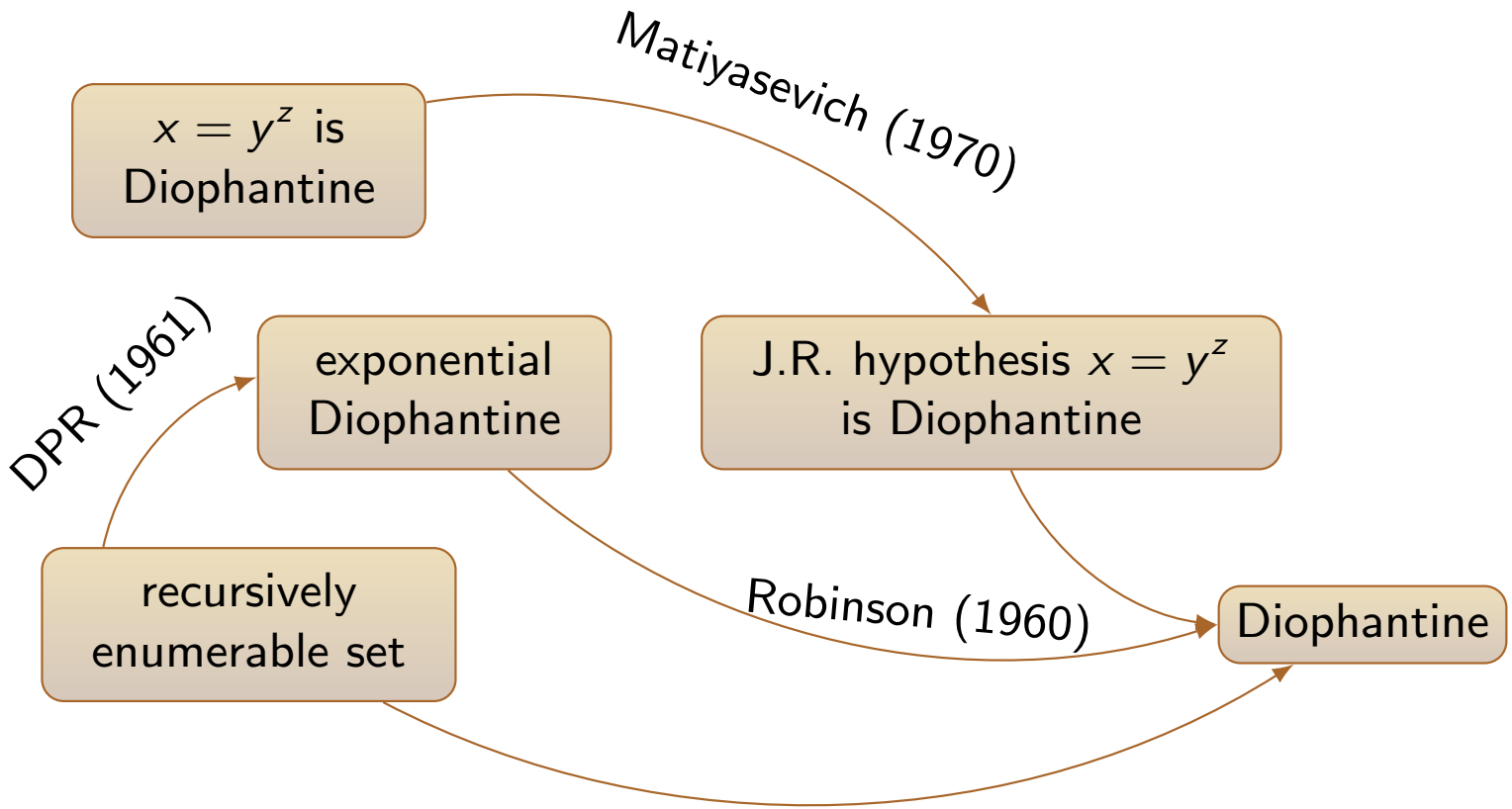
All recursively enumerable sets are Diophantine.

Historical overview

Davis approach (≈1949)

- Hilbert's tenth problem has a negative solution, if
- there is no general algorithm to determine whether a Diophantine equation has solutions in the integers, if
- there exists a Diophantine set that is not recursive, if
- Davis's conjecture is true.

Historical overview



Davis's conjecture (1948)

Normal form for recursively enumerable sets (Martin Davis, 1949)

$$\{a \mid \exists y \forall k \leqslant y \exists x_1, \ldots, x_n : p(a, k, y, x_1, \ldots, x_n) = 0\}$$

Matiyasevich's theorem

Exponentiation is Diophantine i.e. there exists a suitable polynomial P with the property:

$$p = q^r \iff \exists x_1, x_2, \ldots, x_m : P(p, q, r, x1, x2, ..., xm) = 0.$$

The proof technique

- The original approach: From a Diophantine definition of the relation $y = F_{2x}$ where F_0, F_1, F_2, ... are Fibonacci numbers.
- Post–Matiyasevich approach Explores the Pell equation: $x^2 - Dy^2 = 1$.

Selected solutions of Pell's equation

List of the smallest non zero solution to $x^2 - Dy^2 = 1$ for given D:

- $x^2 - 46y^2 = 1$, pair $\langle 24335, 3588 \rangle$.
- $x^2 - 53y^2 = 1$, pair $\langle 66249, 9100 \rangle$.
- $x^2 - 61y^2 = 1$, pair $\langle 1766319049, 226153980 \rangle$.
- $x^2 - 73y^2 = 1$, pair $\langle 2281249, 267000 \rangle$.

Question: Is there always a solution?

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Formalizing 100 Theorems (Freek Wiedijk)

39. Solutions to Pell's Equation
    HOL Light, John Harrison
    Mizar, Marcin Acewicz & Karol Pak
    Metamath, Stefan O'Rear

Pell's equation in Mizar

$N 39. Solutions to Pell's Equation

```
theorem :: PELLS_EQ:14
  D is non square implies
    ex x,y be Nat st x^2 - D * y^2 = 1 & y <> 0;
```

$N The Cardinality of the Pell's Solutions

```
theorem :: PELLS_EQ:17
  for D be non square Nat holds
    the set of all ab where ab is positive
      Pell's_solution of D is infinite;
```

Post-Matiyasevich approach

- Based on a special case (easy case) of the Pell's equation that has the form $x^2 - (a^2 - 1)y^2 = 1$.
- Solutions of the case can be ordered in two sequences recursively:

$$\begin{aligned} x_0(a) &= 1 \\ y_0(a) &= 0 \\ x_{n+1}(a) &= a \cdot x_n(a) + (a^2 - 1) \cdot y_n(a) \\ y_{n+1}(a) &= x_n(a) + a \cdot y_n(a) \end{aligned}$$

Post-Matiyasevich approach - First *advanced* lemma

theorem :: HILB10_1:38
  for y,z,a be Nat holds
    y = Py(a,z) & a > 1 iff

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```
theorem :: HILB10_1:38
  for y,z,a be Nat holds
      y = Py(a,z) & a > 1 iff
  ex x,x1,y1,A,x2,y2 be Nat st
    a>1 &
    [x,y] is Pell's_solution of (a^2- 1) &
    [x1,y1] is Pell's_solution of (a^2- 1) &
    y1>=y & A > y & y >= z &
    [x2,y2] is Pell's_solution of (A^2- 1) &
    y2,y are_congruent_mod x1 &
    A,a are_congruent_mod x1 &
    y2,z are_congruent_mod 2*y &
    A,1 are_congruent_mod 2*y &
    y1,0 are_congruent_mod y^2;
```

Post-Matiyasevich approach - Second *advanced* lemma

```
theorem :: HILB10_1:39
    for x,y,z be Nat holds
        y = x|^z
    iff
      (y = 1 & z = 0) or
      (x = 0 & y = 0 & z > 0) or
      (x = 1 & y = 1 & z > 0) or
      (x > 1 & z > 0 & ex y1,y2,y3,K be Nat st
           y1 = Py(x,z+1) & K > 2*z*y1 &
           y2 = Py(K,z+1) & y3 = Py(K*x,z+1) &
           (0 <= y-y3/y2 <1/2 or 0 <= y3/y2 -y < 1/2));
```

Diophantine set

A Diophantine set is a subset A of $\mathbb{N}^i$ for some i such that there exists j and a polynomial equation with integer coefficients and unknowns $P(x, y) = 0$ with $x \in \mathbb{N}^i$, $y \in \mathbb{N}^j$ such that

$$\forall_{a \in \mathbb{N}^i}\ a \in A \iff \exists_{b \in \mathbb{N}^j} P(a, b) = 0.$$

```
definition
  let n be Nat;
  let A be Subset of n -xtuples_of NAT;
  attr A is diophantine means :: HILB10_2:def 6
    ex m being Nat,p being INT-valued Polynomial of n+m,F_Real st
      for s holds
            s in A iff ex x being n-element XFinSequence of NAT,
                       y being m-element XFinSequence of NAT st
                         s = x & eval(p,@(x^y)) = 0;
end;
```

Matiyasevich's piece of the puzzle

theorem HILB10_3:23
for i1,i2,i3 be Element of n holds
{p where p be n-element XFinSequence of NAT:
p.i1 = Py(p.i2,p.i3) & p.i2 > 1}
is diophantine Subset of n -xtuples˙of NAT

theorem HILB10_3:24
for i1,i2,i3 be Element of n holds
{p where p be n-element XFinSequence of NAT:
p.i2 =(p.i1) |^ (p.i3)}
is diophantine Subset of n -xtuples˙of NAT

Proof overview

Intersections and unions of Diophantine sets are Diophantine.

Proof. Suppose $P1(T, X)$, $P2(T, Y)$ are polynomials that determine subsets $A1$ and $A2$, respectively. Then

$$P1(T, X) \cdot P2(T, Y), \;\; P1(T, X)^2 + P2(T, Y)^2$$

are suitable polynomials to determine $A1 \cup A2$, $A1 \cap A2$.

Substitution

If $R \subset \omega^{n+1}$ is Diophantine and F is an n-ary function with a Diophantine graph, then the relation $S(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ defined by

$$S : R(x_0, \ldots, x_{i-1}, F(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), x_{i+1}, \ldots, x_n)$$

is also Diophantine.

Substitution in Mizar

```
scheme Substitution{P[Nat,Nat,Nat,Nat,Nat,Nat],
  F(Nat,Nat,Nat)→Nat}:
    for i1,i2,i3,i4,i5 holds {p: P[p.i1,p.i2,F(p.i3,p.i4,p.i5),p.i3,p.i4,p.i5]}
      is diophantine Subset of n -xtuples_of NAT
provided
  A1: for i1,i2,i3,i4,i5,i6 holds {p: P[p.i1,p.i2,p.i3,p.i4,p.i5,p.i6]}
    is diophantine Subset of n -xtuples_of NAT
and
  A2:for i1,i2,i3,i4 holds {p: F(p.i1,p.i2,p.i3) = p.i4}
    is diophantine Subset of n -xtuples_of NAT
```

DPRM Theorem – current state

Probably one of the last advanced lemma

If $R \subset \omega^{n+2}$ is Diophantine then $\forall_{y \leqslant x}\{a | a^\frown \langle x, y\rangle \in R\}$ is Diophantine.

DPRM Theorem – current state

Probably one of the last advanced lemma

If $R \subset \omega^{n+2}$ is Diophantine then $\forall_{y \leqslant x}\{a | a^\frown\langle x, y\rangle \in R\}$ is Diophantine.

2.11. Bounded Quantifier Theorem. *Let* $P(A_0, ..., A_{m-1}, X, Y, X_0, ..., X_{n-1})$ *be a polynomial. There is a polynomial* $Q(A_0, ..., A_{m-1}, X)$ *such that, for any* $a_0, ..., a_{m-1} \in \omega$,

i. $\forall x\, [Q(a_0, ..., a_{m-1}, x) \geq x]$

ii. $\forall x\, \forall y x_0 ... x_{n-1} \leq x\, [|P(\boldsymbol{a}, x, y, \boldsymbol{x})| \leq Q(\boldsymbol{a}, x)]$

iii. *for any x, the following are equivalent:*

a. $\forall y \leq x\, \exists x_0 ... x_{n-1} \leq x\, [P(\boldsymbol{a}, x, y, \boldsymbol{x}) = 0]$

b. $$\exists c t v_0 ... v_{n-1}\, [t = Q(\boldsymbol{a},x)! \wedge 1 + (c+1)t = \prod_{m=0}^{x}(1 + (m+1)t) \wedge$$
$$\wedge\ 1 + (c+1)t \mid \prod_{j=0}^{x}(v_0 - j) \wedge ... \wedge 1 + (c+1)t \mid \prod_{j=0}^{x}(v_{n-1} - j) \wedge$$
$$\wedge\ 1 + (c+1)t \mid P(a_0, ..., a_{m-1}, x, c, v_0, ..., v_{n-1})].$$

Proof technique

Chinese remainder theorem, 4 combinatorial lemmas: binomial is Diophantine and further factorial, two cases of product.

Currently up to the 4-th lemma

```
theorem step4:
    for x,y,x1 be Nat st x1 >= 1holds
      y = Product (1+(x1 * idseq x))
    iff
      ex u,w,y1,y2,y3,y4,y5 be Nat st
          u > y & x1*w,1 are˙congruent˙mod u &
          y1 = x1|^x &
          y2 = x! &
          y3 = (w+x) choose x &
          y1*y2*y3,y are_congruent_mod u &
          y4 = 1+x1*x &
          y5 = y4|^x & u > y5
```

Conclusions

- 4 Mizar articles, 8600–lines.
- 1 goal from Freek Wiedijk's list of "Top 100 mathematical theorems".
- Matiyasevich's theorem (piece of the puzzle).