Finding and proving new geometry theorems in regular polygons with dynamic geometry and automated reasoning tools

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August 16, 2018

Abstract

In 1993 Watkins and Zeitlin published a method to simply compute the minimal polynomial of $\cos(2\pi/n)$, based on the Chebyshev polynomials of the first kind. In the present contribution a small augmentation to GeoGebra is shown: GeoGebra is now capable to discover and automatically prove various non-trivial properties of regular n-gons.
Discovering and proving a conjecture can be sketched with GeoGebra, then, in the background a rigorous proof is computed, so that the conjecture can be confirmed, or must be rejected.
Besides confirming well known results, many interesting new theorems can be found, including statements on a regular 11-gon that are impossible to represent with classical means, for example, with a compass and a straightedge, or with origami.

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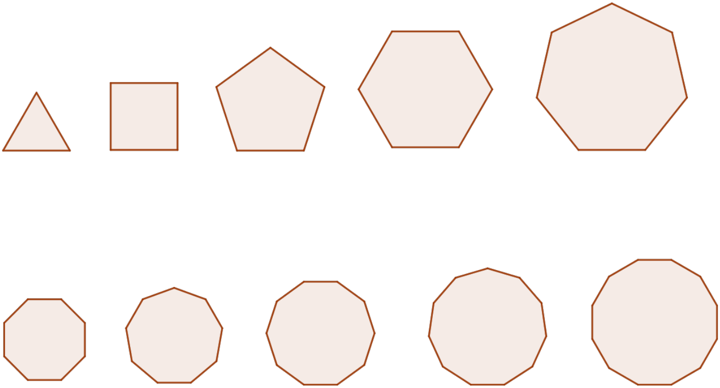

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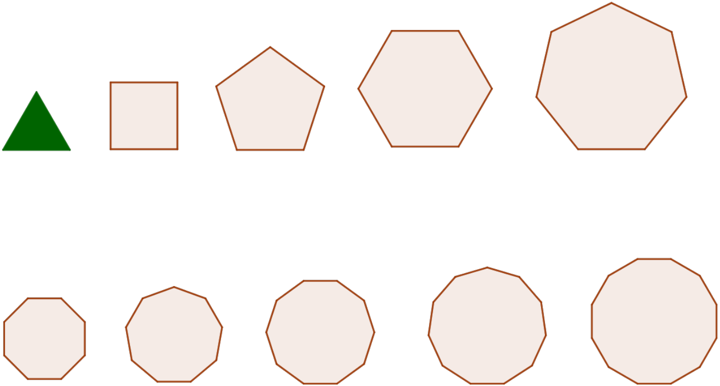

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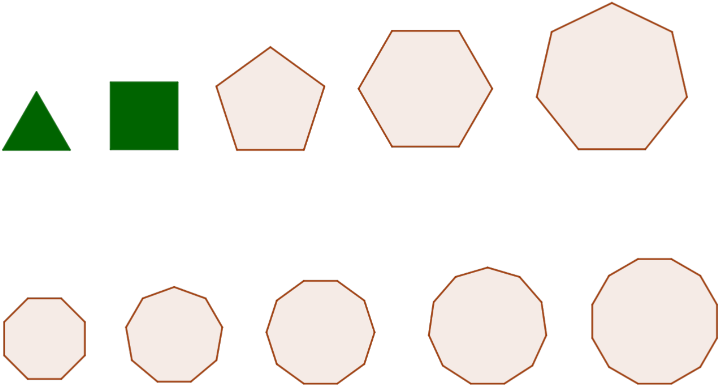

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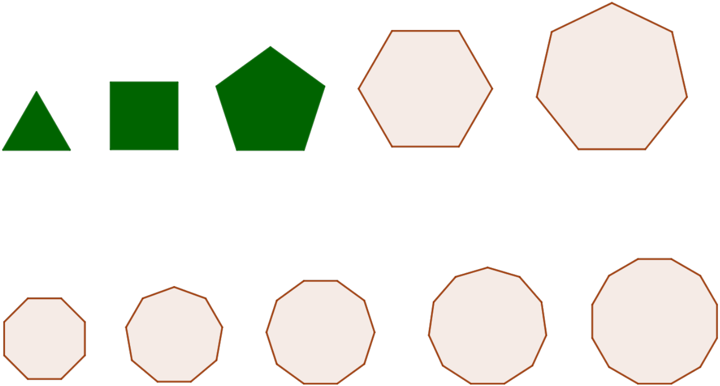

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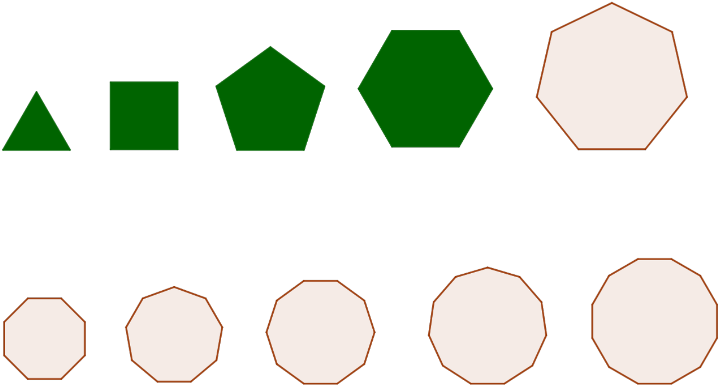

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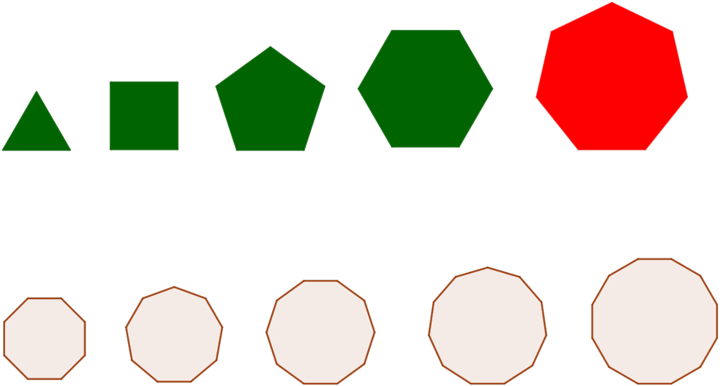

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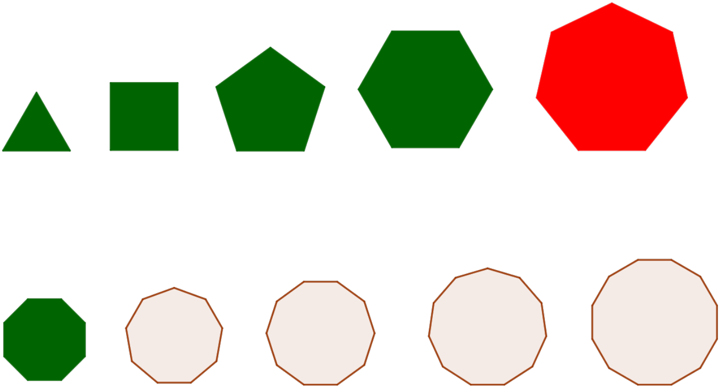

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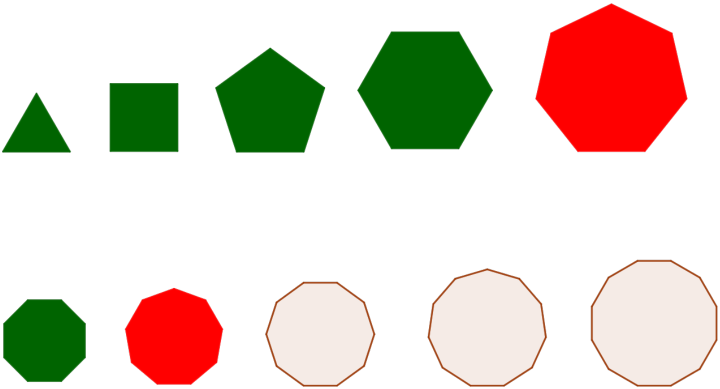

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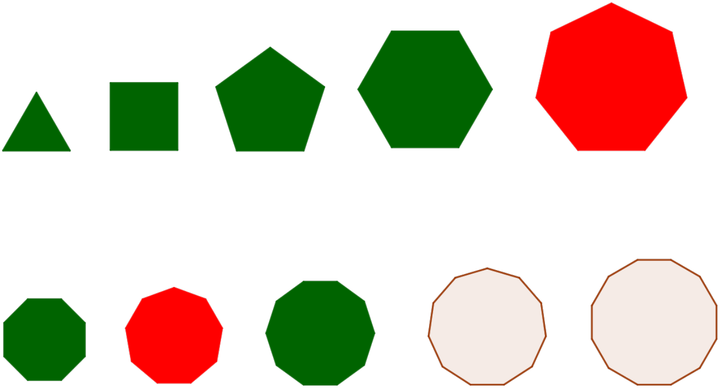

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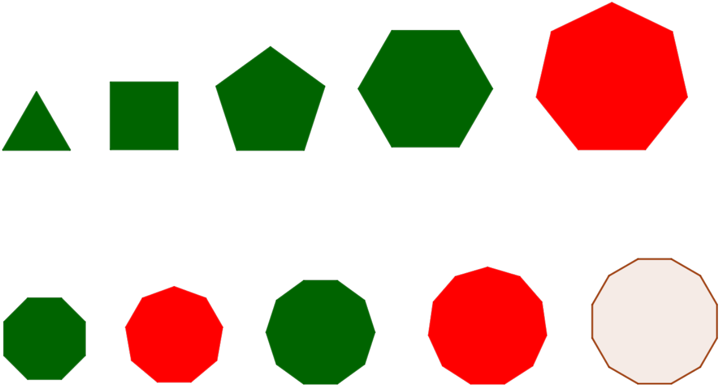

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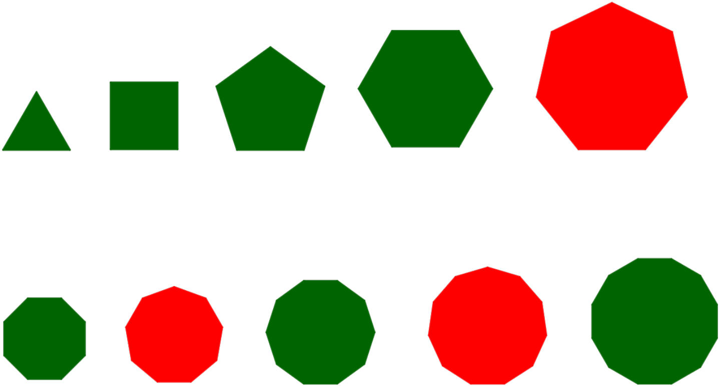
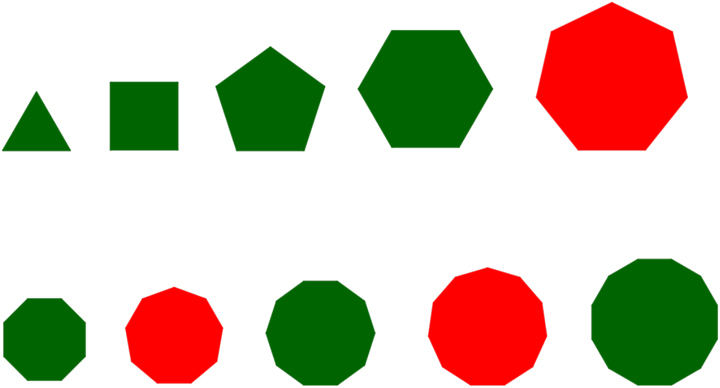


Which regular polygons can be constructed with **origami (paper folding)**?



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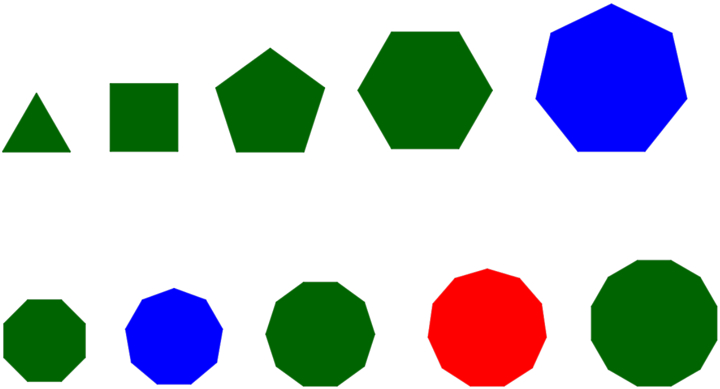

General theorems on constructibility

Theorem (Gauß-Wantzel, 1837)

A regular n-gon is constructible with compass and straightedge if and only if

$$n = 2^k \cdot p_1 \cdot p_2 \cdots p_\ell$$

where the p_i are all different prime numbers such that $p_i - 1 = 2^m$ $(k, \ell, m \in \mathbb{N}_0)$.

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Theorem (Pierpont, 1895)

A regular n-gon is constructible with origami if and only if

$$n = 2^k \cdot 3^r \cdot p_1 \cdot p_2 \cdots p_\ell$$

where the p_i are all different prime numbers such that $p_i - 1 = 2^m \cdot 3^s$ ($k, \ell, m, r, s \in \mathbb{N}_0$).

Consequences

Corollary

A regular 11-gon cannot be constructed with compass and straightedge, or with origami.

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The same statement is valid for $n = 22, 23, 25, 29, 31, \ldots$

Related work

- Theorems on regular n-gons for small n are well known (including theorems in mathematics curriculum), including
  - constructibility theorems (also in primary/secondary school),
  - statements on the golden ratio in regular pentagons.
- Some exotic results are known for bigger n, e.g. for $n = 9$ Karst's statement is known (`https://www.geogebra.org/m/AXd5ByHX#material/x5u93pFr`).
- Mechanical geometry theorem proving is a well known technique, initiated by Wen-Tsün Wu and popularized by his followers, including Chou, and by Kapur, Buchberger, Kutzler and Stifter, Recio and Vélez, and others. Several thousands of theorems can be mechanically proven very quickly—but they are unrelated to regular polygons.

This contribution. . .

- is based on Wu's approach in algebraizing the geometric setup,
- exploits the power of Gröbner basis computations,
- can be further developed towards automated discovery ($\rightarrow$ `RegularNGons`),
- uses a sequence of formulas by Watkins and Zeitlin, based on the Chebyshev polynomials of the first kind (in order to describe consecutive rotations of the edges around one of their endpoints (=a vertex) by using $\cos(2\pi/n)$ and $\sin(2\pi/n)$).

Computing the minimal polynomial of $\cos(2\pi/n)$

Lehmer (1933), Watkins–Zeitlin (1993), recap. Gurtas (2017)

```
procedure Cos2piOverNMinpoly(n)
    p_c ← T_n − 1
    for all j | n ∧ j < n do
        q ← T_j − 1
        r ← gcd(p_c, q)
        p_c ← p_c / r
    return SquarefreeFactorization(p_c)
```

where T_n stands for the n^{th} Chebyshev polynomial of the first kind (see `https://dlmf.nist.gov/18.9` for its recurrence relations).

Minimal polynomial of $\cos(2\pi/n)$

n	Minimal polynomial
1	$x-1$
2	$x+1$
3	$2x+1$
4	x
5	$4x^2+2x-1$
6	$2x-1$
7	$8x^3+4x^2-4x-1$
8	$2x^2-1$
9	$8x^3-6x+1$
10	$4x^2-2x-1$
11	$32x^5+16x^4-32x^3-12x^2+6x+1$
12	$4x^2-3$
13	$64x^6+32x^5-80x^4-32x^3+24x^2+6x-1$
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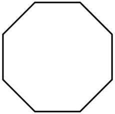

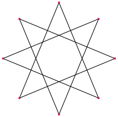

An equation system

describing the vertices of the regular n-gon

Let its vertices be P_i and their coordinates (x_i, y_i) $(i = 0, 1, 2, \ldots, n-1)$.

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$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdot \left(\begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} - \begin{pmatrix} x_{i-2} \\ y_{i-2} \end{pmatrix} \right)$$

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   and therefore

$$x_i = -xy_{i-1} + x_{i-1} + xx_{i-1} + yy_{i-2} - xx_{i-2}, \tag{1}$$

$$y_i = y_{i-1} + xy_{i-1} + yx_{i-1} - xy_{i-2} - yx_{i-2} \tag{2}$$

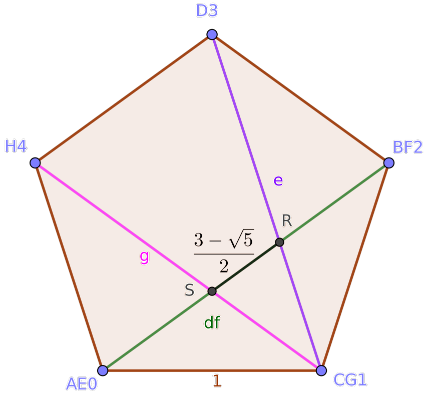
   for all $i = 2, 3, \ldots, n-1$.

Example 1

Lengths in a regular pentagon (a classic result)

Theorem

Consider a regular pentagon with vertices $P_0, P_1, \ldots, P_4$. Let $A = P_0$, $B = P_2$, $C = P_1$, $D = P_3$, $E = P_0$, $F = P_2$, $G = P_1$, $H = P_4$. Let us define diagonals $d = AB, e = CD, f = EF, g = GH$ and intersection points $R = d \cap e, S = f \cap g$. Now, when the length of P_0P_1 is 1, then the length of RS is $\frac{3-\sqrt{5}}{2}$.



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Lengths in a regular pentagon (a classic result, proof)

$$
\begin{aligned}
h_1 &= 4x^2 + 2x - 1 = 0, && \text{(minimal polynomial of } \cos(2\pi/5))\\
h_2 &= x^2 + y^2 - 1 = 0, && \text{(one possible } y \text{ is } \sin(2\pi/5))\\
h_3 &= x_0 = 0, && (x\text{-coordinate of } P_0)\\
h_4 &= y_0 = 0, && (y\text{-coordinate of } P_0)\\
h_5 &= x_1 - 1 = 0, && (x\text{-coordinate of } P_1)\\
h_6 &= y_1 = 0, && (y\text{-coordinate of } P_1)\\
h_7 &= -x_2 - xy_1 + x_1 + xx_1 + yy_0 - xx_0 = 0,\\
h_8 &= -y_2 + y_1 + xy_1 + yx_1 - xy_0 - yx_0 = 0,\\
h_9 &= -x_3 - xy_2 + x_2 + xx_2 + yy_1 - xx_1 = 0,\\
h_{10} &= -y_3 + y_2 + xy_2 + yx_2 - xy_1 - yx_1 = 0,\\
h_{11} &= -x_4 + -xy_3 + x_3 + xx_3 + yy_2 - xx_2 = 0,\\
h_{12} &= -y_4 + y_3 + xy_3 + yx_3 - xy_2 - yx_2 = 0.
\end{aligned}
$$

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Since $R \in d$ and $R \in e$, we can claim that

$$h_{13} = \begin{vmatrix} x_0 & y_0 & 1 \\ x_2 & y_2 & 1 \\ x_r & y_r & 1 \end{vmatrix} = 0, h_{14} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_r & y_r & 1 \end{vmatrix} = 0,$$

where $R = (x_r, y_r)$. Similarly,

$$h_{15} = \begin{vmatrix} x_0 & y_0 & 1 \\ x_2 & y_2 & 1 \\ x_s & y_s & 1 \end{vmatrix} = 0, h_{16} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_s & y_s & 1 \end{vmatrix} = 0,$$

where $S = (x_s, y_s)$. Finally we can define the length $|RS|$ by stating

$$h_{17} = |RS|^2 - \left((x_r - x_s)^2 + (y_r - y_s)^2\right) = 0.$$

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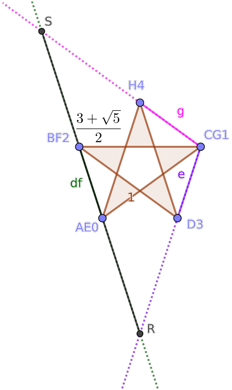

We may want to directly prove that $|RS| = \frac{3-\sqrt{5}}{2}$. This actually does not follow from the hypotheses, because the star-regular pentagon case yields a different result.

That is, we need to prove a weaker thesis, namely that $|RS| = \frac{3-\sqrt{5}}{2}$ or $|RS| = \frac{3+\sqrt{5}}{2}$, which is equivalent to

$$\left(|RS| - \frac{3-\sqrt{5}}{2}\right) \cdot \left(|RS| - \frac{3+\sqrt{5}}{2}\right) = 0.$$

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Unfortunately, this form is still not complete, because $|RS|$ is defined implicitly by using $|RS|^2$, that is, if $|RS|$ is a root, also $-|RS|$ will appear. The correct form for a polynomial t that has a root $|RS|$ is therefore

$$t = \left(|RS| - \frac{3-\sqrt{5}}{2}\right) \cdot \left(|RS| - \frac{3+\sqrt{5}}{2}\right) \cdot \left(-|RS| - \frac{3-\sqrt{5}}{2}\right) \cdot \left(-|RS| - \frac{3+\sqrt{5}}{2}\right) = 0,$$

that is, after expansion,

$$t = (|RS|^2 - 3|RS| + 1) \cdot (|RS|^2 + 3|RS| + 1) = |RS|^4 - 7|RS|^2 + 1 = 0.$$

Now, finally, the proof will be performed by showing the negation of t. This is accomplished by adding $t \cdot z - 1 = 0$ to the equation system $\{h_1, h_2, \ldots, h_{17}\}$ and obtaining a contradiction. □

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→ `https://github.com/kovzol/RegularNGons`

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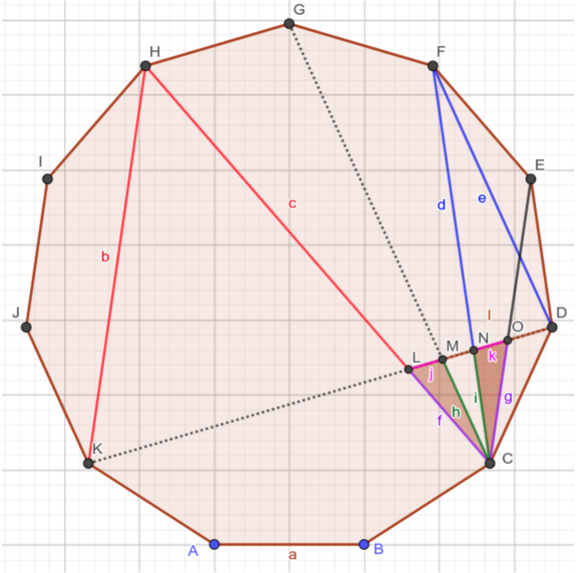


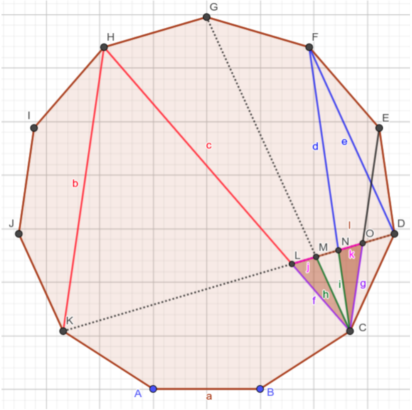
Example 2

Lengths in a regular 11-gon

Theorem

A regular 11-gon (having sides of length 1) is given. Then:

1. $b = c$,
2. $d = e$,
3. *triangles CLM and CON are congruent,*
4. $a = l$ (*that is,* $|AB| = |DL|$).
5. *Let* $P = BJ \cap CD$. *Then* $|OP| = \sqrt{3}$.
6. $|BO| \neq \frac{5}{3}$ (*but it is very close to it,* $|BO| \approx 1,66686\ldots$, *it is a root of the polynomial* $x^{10} - 16x^8 + 87x^6 - 208x^4 + 214x^2 - 67 = 0$).



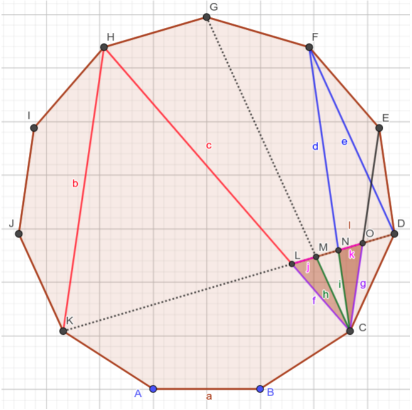
Example 2

Lengths in a regular 11-gon

Theorem

A regular 11-gon (having sides of length 1) is given. Then:

1. $b = c$,
2. $d = e$,
3. *triangles CLM and CON are congruent,*
4. $a = l$ *(that is,* $|AB| = |DL|$*).*
5. *Let* $P = BJ \cap CD$*. Then* $|OP| = \sqrt{3}$.
6. $|BO| \neq \frac{5}{3}$ *(but it is very close to it,* $|BO| \approx 1{,}66686\ldots$*, it is a root of the polynomial* $x^{10} - 16x^8 + 87x^6 - 208x^4 + 214x^2 - 67 = 0$*).*



https://www.geogebra.org/m/AXd5ByHX#material/YVTKjR2E

Implementation in GeoGebra

. . . and further results

https://www.geogebra.org/m/AXd5ByHX

Implementation in GeoGebra

…and further results

`https://www.geogebra.org/m/AXd5ByHX`

In case you get no answer when clicking "More…", consider a second try. There is a bug in GeoGebra—sometimes the computer algebra system is not loaded automatically on the web.

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A workaround: https://kovz0l.blogspot.com/2018/05/preloading-cas-in-geogebra-applets.html.

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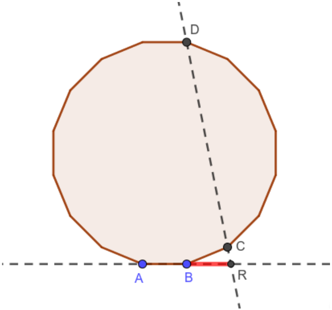
See also https://github.com/kovzol/gg-art-doc/blob/master/pdf/english.pdf for a tutorial on *GeoGebra Automated Reasoning Tools* and http://www.researchinformation.co.uk/timearch/2018-02/pageflip.html on *Using Automated Reasoning Tools in GeoGebra in the Teaching and Learning of Proving in Geometry* by K., Recio and Vélez (2017, 2018).

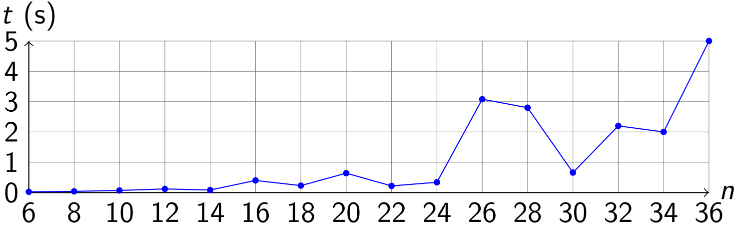
How fast is it?

A simple theorem for benchmarking

Theorem

Let n be an even positive number ($n \geq 6$), and let us denote the vertices of a regular n-gon by $P_0, P_1, \ldots, P_{n-1}$. Let $A = P_0$, $B = P_1$, $C = P_2$, $D = P_{n/2}$. Moreover, let $R = AB \cap CD$. Then $|AB| = |BR|$.





Conclusion

- A method that helps obtaining various new theorems on regular polygons, based on the work of Wu (1984), Watkins–Zeitlin (1993) and Recio–Vélez (1999)
- Manual search
- GeoGebra implementation (based on Gröbner bases via the Giac CAS)
- The software tool `RegularNGons` finds theorems automatically by elimination
  - a work in progress on approximating π is available at `https://arxiv.org/abs/1806.02218`

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Thank you for your kind attention!